The coriolis-effect in meterology

Mats Rosengren

29.12.2018

Because of its rotation the Earth was formed in a slightly flattend shape. This shape is such that the gravitational force at any point of the Earth surface is slightly tilted away from the equator as displayed in the figure below.

The force component in the local equatorial plane caused by this tilt has exactly the strength required to prevent that the Earth rotation makes a "mass element" to slip away towards the equator. As long as the atmosphere is at rest relative to the Earth surface underneath the equatorial component of the gravitational force has the same strength as the equatorial component of the "centrifugal force" and the atmosphere is at an "equilibrium state" relative to the rotating Earth. But if the atmosphere is put in motion, i.e. if there is wind, the balance between the gravitational force is disturbed and an north/south oscillation around the equilibrium will result. Because of the different rotational velocities of the Earth surface at different latitudes this oscillation will take the form of "whirls" relative to the Earth surface! For the detailed investigation of the motion of the atmosphere it is convenient to introduce a reference-system rotating with the Earth.

The equation of motion in a reference system rotating with a constant rate $\bar{\omega}$ relative inertial space takes the form

$$\bar{a} = -2\ \bar{\omega} \times\ \bar{v} + \omega^2 \bar{r}_{ort}\ + \bar{f}_1 + \bar{f}_2$$

where

$\bar{a}$ is the acceleration relative to the rotating reference system

$\bar{\omega}$ is the rotation vector of the rotating system relative to inertial space

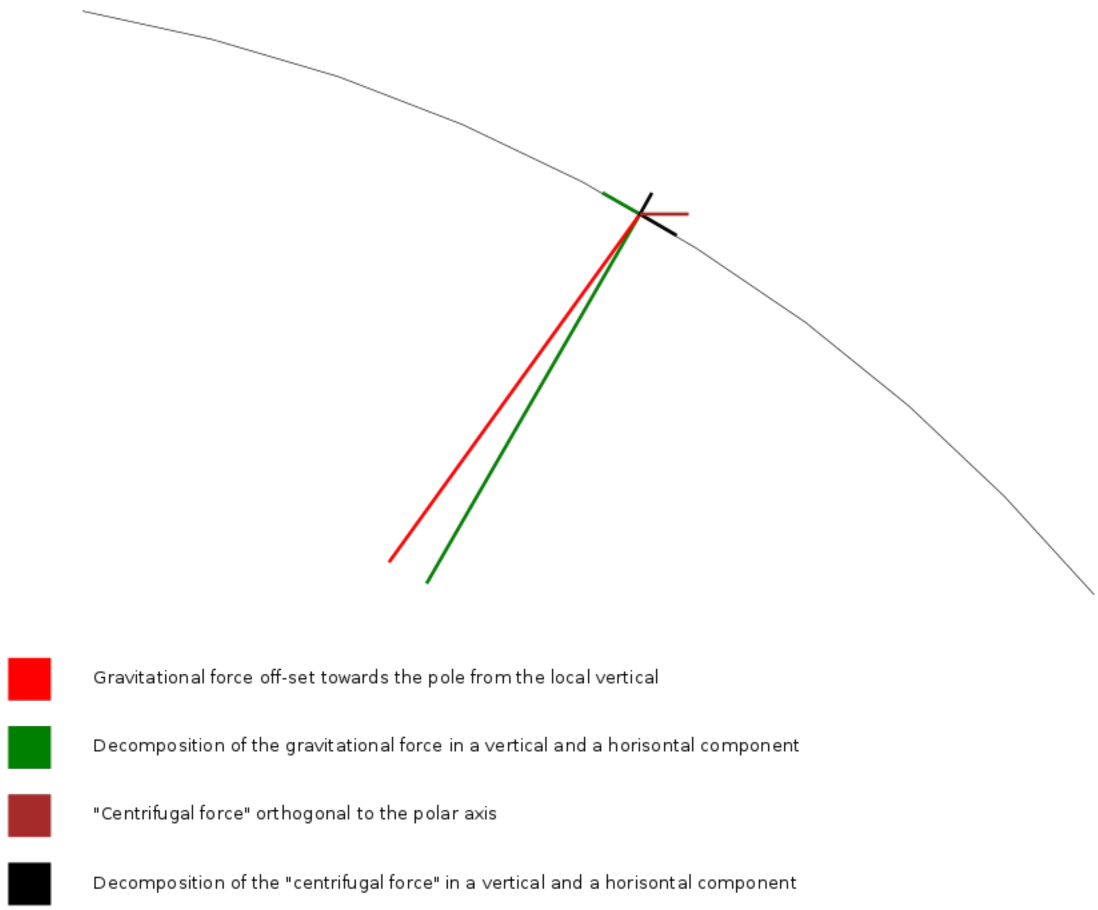
$\bar{v}$ is the velocity relative to the rotating reference system

ω is the (scalar) rotation rate of the rotating system relative to inertial space

$\bar{r}_{ort}$ is the projection of the position vector $\bar{r}$ on the plane orthogonal to $\bar{\omega}$

$\bar{f}_1$ is the gravitational force (per unit mass)

$\bar{f}_2$ is the force (per unit mass) from the ground below on which the mass is sliding



The Earth has an oblate shape such that the horisontal component of the gravitational force prevents a mass point at rest on the surface to slide away towards the equator

When the atmosphere moves with the ground underneath, i.e. when there is no wind, $\bar{v}$ is equal to zero and the acceleration relative to the rotating reference system is

$\bar{a} = \omega^2 \bar{r}_{ort} \ + \bar{f}_1 + \bar{f}_2$

This is the situation displayed in the figure. The first term $\omega^2 \bar{r}_{ort}$ is the "centrifugal force" indicated in the figure with brown color. The second term $\bar{f}_1$ is in the gravitational force indicated with red colour. The third term $\bar{f}_2$ is the vertical force from the ground below carrying the mass counteracting the sum of the vertical components of the gravitational force(green) and the "centrifugal force" (black) As $\bar{f}_1$ does not depend on the velocity of the mass relative to the Earth and as $\bar{f}_2$ is orthogonal to the surface the acceleration of the "sliding motion" over the Earth surface $\bar{a}_{equ}$ is given by

$\bar{a}_{equ} \ = -2 \ [\bar{\omega} \times \ \bar{v}]_{equ}$

To find the acceleration of a mass (air!) sliding over the Earth surface at latitude ϕ, introduce a coordinate system in the longitudal plane of the point with z-axis in the direction of the Earth axis and x-axis in the longitudal plane such that the outward normal is $\hat{u} \ = (\cos\phi \ , \ 0 \ , \ \sin\phi)$ Direction "east" at this point is then $\hat{e} \ = \ (\ 0 \ , \ 1 \ , \ 0 \ )$ and direction "north" is $\hat{n} \ = \ (-\sin\phi \ , \ 0 \ , \cos\phi)$ The vectors $\hat{e}$ and $\hat{n}$ are then spanning the horisontal plane and any possible velocity for the (air) mass is of the form

$v \ (\ \hat{e} \ \cos\alpha \ + \hat{n} \ \sin\alpha \ )$

for some angle α

It follows that

$\bar{\omega} \times \bar{v} \ = \ \omega \ v \ (\ -\cos\alpha \ , \sin\phi \sin\alpha \ , \ 0 \ )$

The direction "right" in the equatorial plane is $\hat{r} \ = \ (\ \sin\phi \ cos\alpha \ , \ \sin\alpha \ , \ -\cos\phi \ cos\alpha \ )$
and one has that
$< \ \hat{u} \mid \bar{\omega} \times \bar{v} \ > = -\cos\phi \cos\alpha$
$< \ \hat{r} \mid \bar{\omega} \times \bar{v} \ > = \sin\phi$

The component of the "coriolis force" in the direction "upwards" is consequently $\bar{f}_{up} \ = \ \omega \, v \, 2 \ \cos\phi \cos\alpha \ \hat{u}$ and the component in the direction "right" is $\bar{f}_{right} \ = \ \omega \ v \ 2 \ \sin\phi \ \ \hat{r}$

The force component $\bar{f}_{up}$ does not affect the motion as it is assumed that the mass point is constraint to "slide" along the Earth surface.

For this "sliding motion" on the Earth surface the equation of motion is

$\bar{a}_{equ} \ = \ \omega \ v \ 2 \ \sin\phi \ \hat{r}$

where

$\bar{a}_{equ}$ is the acceleration
ω is the rotation rate of the Earth
v is the velocity relative to the rotating Earth
ϕ is the latitude

$\hat{r}$ is the unit vector in the local horisontal plane that is orthogonal to the velocity vector and is pointing "right" relative the direction of motion

In the northern hemisphere ϕ is positive and the direction of the acceleration is towards "right". In the southern hemisphere ϕ is negative and the direction of the acceleration is towards "left"

Neglecting the variation of the latitude ϕ this differential equation defines the mass point moving with a constant velocity uniformly curving to the right (in the southern hemisphere to the left). On a surface of a sphere this corresponds to a "small circle". But because of the factor $\sin\phi$ the curvature is not uniform but the "bending" is stronger the further from the equator one gets, the result is then rather a "spiral"

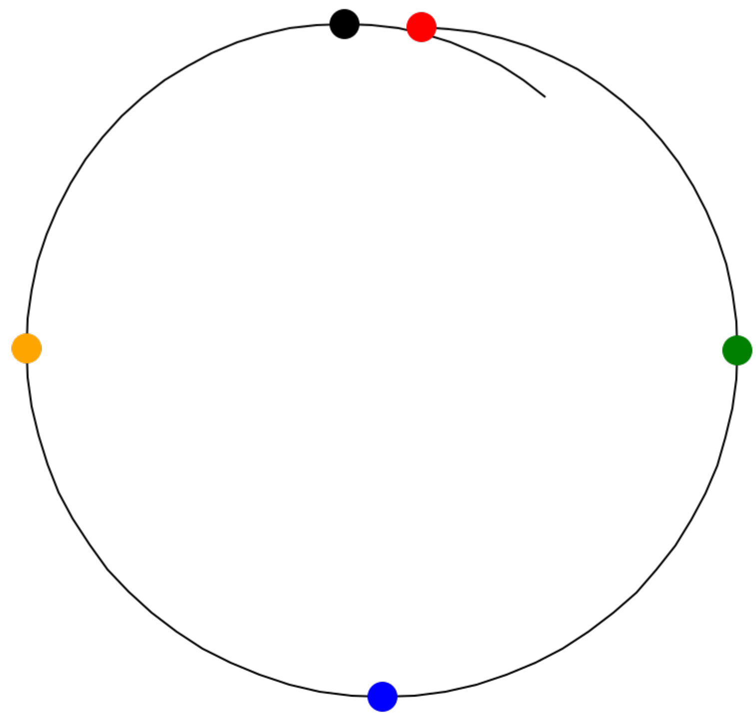
A solution to the equation of motion above for a mass point sliding on the Earth surface is displayed in the next figure!

Initially the mass point is moving with a velocity of 50 m/s over the Earth surface, this is the red point. As its eastward velocity is higher then that of the Earth surface the gravitational component tangential to the Earth surface is here too weak to prevent the mass point from starting to slide southward towards the equator. At its path towards south the local velocity of the Earth surface gets higher and at the green point the eastward velocity of the mass point is the same as that of the local Earth surface. But here the mass point has picked up speed towards south and this southward movement continues and only comes to a halt at the blue point. At this point the Earth surface moves 50 m/s faster towards east then the mass point. As now the tangential component of the gravitation is higher then what is needed to prevent the mass point from sliding towards the equator the mass point is instead accelerated towards north until it has reached the yellow point. But at this time the mass has picked up enough speed to continue further north to the black point and everything starts over again from the beginning.

As the acceleration of the motion relative the Earth surface is orthogonal to its direction the velocity relative the Earth surface stays constant. But this is not the case for the velocity relative to inertial space. The tangential component of the gravitational force causes the kinetic energy of the mass point to increase when the mass point moves away from the equator. As this force is in the longitudinal plane there is no change in angular momentum relative the Earth axis, though.

The following table is applicable for the figure below

Latitude(deg N)	Radius(km)	Surface Inertial Velocity(km/s)	Air Inertial Velocity(km/s)
50.0	4108	0.2996	0.2996 + 0.050 = 0.3496
41.4	4791	0.3494	0.3494 - 0.050 = 0.2994



Latitude 50.0 deg North velocity 50 m/s towards East

3h59m later. Latitude 45.7 deg North, Longitude + 5.8 deg , velocity 50 m/s towards South

8h21m later. Latitude 41.4 deg North, Longitude - 0.7 deg , velocity 50 m/s towards West

12h44m later. Latitude 45.7 deg North, Longitude - 7.3 deg , velocity 50 m/s towards North

16h44m later. Latitude 50.0 deg North, Longitude - 1.5 deg , velocity 50 m/s towards East

A solution to the equation of motion for a mass point sliding on the Earth surface